

# 2nd-order symmetric Lorentzian manifolds I: characterization and general results

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**Abstract.** The  $n$ -dimensional Lorentzian manifolds with vanishing second covariant derivative of the Riemann tensor — 2-symmetric spacetimes — are characterized and classified. The main result is that either they are locally symmetric or they have a covariantly constant null vector field, in this case defining a subfamily of Brinkmann's class in  $n$  dimensions. Related issues and applications are considered, and new open questions presented.

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## 1. Introduction

The aim of this paper is to characterize the manifolds  $\mathcal{V}$  with a metric  $g$  of Lorentzian signature such that the Riemann tensor  $R^\alpha_{\beta\gamma\delta}$  of  $(\mathcal{V}, g)$  *locally* satisfies the second order condition

$$\nabla_\mu \nabla_\nu R^\alpha_{\beta\gamma\delta} = 0. \quad (1)$$

It is quite surprising that, hitherto, despite their simple definition, this type of Lorentzian manifolds have been hardly considered in the literature. Probably this is due to a combination of reasons of diverse origin, such as:

- the classical Riemannian result [53, 56, 81] according to which all such spaces are actually locally symmetric [17] if the metric is positive definite, see section 2;
- even in cases with other signatures for the metric, there are well-known results [81, 26] restricting the possibility of (1). Thus, one can say that manifolds with the property (1) are somehow exceptional: see next section for a recollection of such results and some new expanded ones along these lines.
- characterizing the exceptional cases left over by the previous considerations turns out to be, if not difficult, very laborious indeed. Given that one knows beforehand that the result will be a very special class of metrics this may have prevented some researchers from considering the problem. It must be remarked, nevertheless, that the 4-dimensional case is easily solved —see the footnote number 1 for the sketch of the solution!

The interest of these spacetimes comes also from different perspectives and fields. To start with, they have an obvious mathematical interest. Having such a simple and natural definition, they should be identified for arbitrary signature. Furthermore, there are direct geometric interpretations of condition (1) in analogy with the case of a vanishing first covariant derivative [58]: the tensor field  $\nabla_\nu R^\alpha_{\beta\gamma\delta}$  is

invariant under parallel displacement, and therefore one can say that the curvature is locally a “linear” function of appropriate coordinates. This has a relation to the local holonomy group of the manifold.

In a more concrete manner, (i) the condition (1) implies that given any geodesic curve on the manifold with tangent vector  $\vec{v}$ , and parallelly propagated vector fields  $\vec{X}, \vec{Y}, \vec{Z}$  along the geodesic, then the vector field

$$\nabla_{\vec{v}} R(\vec{X}, \vec{Y}) \vec{Z}$$

is itself parallelly propagated along the geodesic. One could say, leaving aside rigour, that the vector field  $R(\vec{X}, \vec{Y}) \vec{Z}$  is a “linear” function on the affine parameter of the geodesics. And similarly, (ii) take the sectional curvature  $\kappa$  of the manifold [58] at any point relative to the tangent planes at that point. Let the tangent planes be parallelly propagated along geodesics, then the function  $\nabla_{\vec{v}} \kappa$  remains constant.

Nevertheless, there is an important difference between (1) and its first order version with only one covariant derivative: in the latter case, it is known that there is (locally) a geodesic symmetry [58], that is to say, there is an isometry  $\varphi$  of the Lorentzian manifold which acts on the tangent spaces as  $d\varphi = -\text{Id}$ . In the case of (1), there is nothing of this sort in general, as one would need an isometry whose *square*, when acting on the tangent spaces, were not proportional to  $\text{Id}$ . But this is impossible in general, even in 2-dimensional cases.

Having considered their immediate geometric interpretation, the spaces satisfying (1) may also have important applications in physics, and in particular in theories concerning gravitation whose classical arena is a Lorentzian manifold. Just to cite a few possible applications or areas of potential interest, let us mention the following:

- they are of interest in the branch of invariant classification of Lorentzian manifolds, and of the “equivalence problem” [75], that is, to decide if two given spacetimes are locally the same.
- in analogy with flat or conformally flat spacetimes, which can be characterized by the vanishing of the Bel or the Bel-Robinson tensors [73] respectively, the spacetimes with a vanishing “(super)<sup>2</sup>-energy” have  $\nabla_{\nu} R^{\alpha}_{\beta\gamma\delta} = 0$ , and those with zero “(super)<sup>3</sup>-energy” will satisfy (1). Therefore, these spacetimes can shed some light into the potentially high number of conserved quantities that one can form with those tensors, see [73], and whether the superenergy construction stops effectively at some level.
- these spacetimes may arise whenever expansions in normal coordinates are used, or relevant. The higher order terms in the expansions contain summands proportional to the higher order derivatives of the Riemann tensor. Therefore, if these derivatives happen to vanish the expansions may become more manageable. Similarly, if one considers the local form of a Lorentzian manifold around any null geodesic (the so-called “Penrose limits” [61]) the various levels of approximation contain summands corresponding to higher-order conditions of type (1). It is worth mentioning here that the main result obtained in this paper will actually imply that many of the non-locally-symmetric spacetimes satisfying (1) will actually be of Penrose limit type: a plane wave [35, 61, 75].
- it will also turn out that they are closely related to spacetimes with special properties concerning its curvature invariants. Many curvature scalar invariants will be zero or constant, linking these spaces to some families of spacetimes which have become a focus of recent interest, see [65, 24, 22] and references therein.
- related to the previous point, they can be of relevance as solutions in higher order Lagrangian physical theories including gravity, as then only a finite number of terms are relevant. In particular, they can be examples of exact solutions (for the background) in string theory when supported by appropriate matter contents, see e.g. [48, 65, 21] and references therein.
- actually, they can provide examples of exact solutions for backgrounds for 11-dimensional supergravity and relatives via M-theory [38, 7]. This is in fact interrelated with the Penrose limits mentioned previously [43, 7, 8].

- in the previous point as well as in general solitonic or black-hole solutions of supersymmetric theories, these solutions must have a covariantly constant spinor, leading to covariantly constant null vector or tensor fields, see [38]. These will be seen to arise naturally in the spacetimes satisfying (1).
- they can provide some examples of solutions in field equations for which the counter-terms regularizing quantum fluctuations are vanishing [41].
- they also seem to be related to what has been called the “ $\epsilon$ -property” [60], which refers basically to the possibility that some components of the Riemann tensor (and its derivatives) can be made as small as desired by a judicious choice of basis.

Observe that in proper Riemannian spaces (i.e., with a positive-definite metric), the vanishing of the square of a tensor implies that the tensor itself is zero. Though this is not so for the semi-Riemannian case, it will be shown that, *in the case of Lorentzian signature*, the use of causal tensors and superenergy techniques, see [73, 6], provides a valid and productive alternative and similar results can be found. I will make use of this in several places.

The main result of this paper is that all Lorentzian manifolds satisfying (1) but with a non-vanishing first derivative of the Riemann tensor must necessarily have a null covariantly constant vector field. Therefore, all these spacetimes belong to a general class of Lorentzian manifolds known as the Brinkmann class [10]. As a matter of fact, this result can be obtained rather quickly in four dimensions by using spinors,<sup>‡</sup> but as we will see it is far from obvious in higher dimensions.

Of course, the fundamental reason behind this result is the now known and better understood degenerately decomposability [85, 86, 87, 46] of the spacetime in this case, which was completely analyzed in [16] for the Lorentzian case, and has been a subject of recent interest with many interesting applications is supergravity and string theory, see e.g. [38, 11, 39, 7, 3] and references therein. This will be briefly analyzed in subsection 3.1.

Throughout,  $\mathcal{V}$  will denote a differentiable manifold and  $g$  a metric tensor of sufficient differentiability. When considering Lorentzian signature, the choice of signature will be  $(-, +, \dots, +)$ , so that timelike vectors have a negative length, spacelike vectors a positive one, and the null vectors, which include the zero vector, have vanishing length. Lorentzian manifolds are assumed to be time orientable with a chosen future direction. Given the type of formulas needed in the paper I have preferred to write most of the calculations and expressions using index notations. It should be clearly understood, however, that no particular basis has been chosen, and the results are general. They can of course be rewritten in index-free form if desired.

## 2. Generalizations of symmetric spaces. Generic cases

Semi-Riemannian (or pseudo-Riemannian) manifolds satisfying (1) constitute an obvious generalization of the well-known locally *symmetric* spaces which satisfy

$$\nabla_\mu R^\alpha{}_{\beta\gamma\delta} = 0 \tag{2}$$

and were introduced, largely studied and classified by É. Cartan [17] in the proper Riemannian case, see e.g. [18, 51, 47], and later in [13, 16, 14, 12] for the Lorentzian case using results from [5]. The general semi-Riemannian cases were treated in e.g. [15, 58], but these results seem to be incomplete, see the recent studies in [50] and references therein. Locally symmetric spaces are themselves generalizations of the constant curvature spaces and, as a matter of fact, there is a natural hierarchy of conditions, shown in Table 1, that can be placed on the curvature tensor. In that table, the restrictions on the curvature

<sup>‡</sup> Indeed, if the curvature satisfies (1), using the Ricci identities for the curvature spinors —formulae (4.9.13-15) in [62]— one easily derives that the Weyl spinor satisfies the condition appearing at the beginning of page 261 in [62], leading to a type N Weyl tensor. The vanishing covariant derivative of the unique principal null direction follows then in various possible ways, for example using the superenergy tensors of [73].

**Table 1.** The hierarchy of conditions on the Riemann tensor

$R^\alpha_{\beta\gamma\nu} \propto \delta^\alpha_\gamma g_{\beta\nu} - \delta^\alpha_\nu g_{\beta\gamma}$	$\nabla_\mu R^\alpha_{\beta\gamma\delta} = 0$	$\nabla_\mu \nabla_\nu R^\alpha_{\beta\gamma\delta} = 0$	$\nabla_{[\mu} \nabla_{\nu]} R^\alpha_{\beta\gamma\delta} = 0$
constant curvature	symmetric	2-symmetric	semi-symmetric

tensor decrease towards the right and each class is strictly contained in the following ones. Thus, all constant curvature manifolds are obviously symmetric but the converse is not true, and analogously for the other cases. The table has been stopped at the level of semi-symmetric spaces, defined by the condition  $\nabla_{[\mu} \nabla_{\nu]} R^\alpha_{\beta\gamma\delta} = 0$ , where round and square brackets enclosing indices indicate symmetrization and antisymmetrization, respectively. As a matter of fact, semi-symmetric spaces were introduced also by Cartan [18] and studied in [77, 78] as the natural generalization of symmetric spaces for the proper Riemannian case, see also [9] and references therein; in the 4-dimensional Lorentzian case they were studied in [49], and also in e.g. [32, 28, 27, 44] and references therein. Explicit proof that there exist non-symmetric semi-symmetric spaces was given in [79].

Two obvious questions arise:

- (i) why is semi-symmetry considered to be the natural generalization of local symmetry, instead of (1)?
- (ii) why not go on further to higher derivatives of the Riemann tensor?

The answer to both questions is actually the same: a classical theorem [53, 56, 81] states that in any proper Riemannian manifold

$$\nabla_{\mu_1} \dots \nabla_{\mu_k} R^\alpha_{\beta\gamma\delta} = 0 \quad \Longleftrightarrow \quad \nabla_\mu R^\alpha_{\beta\gamma\delta} = 0 \quad (3)$$

for any  $k \geq 1$  so that, in particular, (1) is strictly equivalent to (2) in proper Riemannian spaces.

This may also be the reason of why there seems to be no name for the condition (1) in the literature. However, an analogous condition has certainly been used for the so-called recurrent spaces: if there exists  $A_{\mu_1 \dots \mu_k}$  such that  $\nabla_{\mu_1} \dots \nabla_{\mu_k} R^\alpha_{\beta\gamma\delta} = A_{\mu_1 \dots \mu_k} R^\alpha_{\beta\gamma\delta}$  then the space is called  $k$ -recurrent (e.g. [80, 26]), in particular second order recurrent (or 2-recurrent) for  $k = 2$  [53, 68, 82, 84, 54, 80, 25] and recurrent for  $k = 1$  (see e.g. [71] and references therein). Thus, I will call the spaces satisfying (1) *second-order symmetric*, or in short *2-symmetric*, and more generally  $k$ -symmetric when the left condition in (3) holds. The whole class of  $k$ -symmetric spacetimes for all  $k > 1$  has been called “higher order symmetric spaces” recently in [60].

### 2.1. Results at generic points

As a matter of fact, the equivalence (3) holds as well in “generic” cases of semi-Riemannian manifolds of any signature. For some results on this one can consult [81, 26]. A typical reasoning would be as follows. Assume that the left side of (3) holds for either  $k = 2, 3$ , then the Ricci identity applied to  $\nabla_{[\lambda} \nabla_{\mu]} \nabla_\nu R_{\alpha\beta\gamma\delta}$  provides

$$R^\rho_{\nu\lambda\mu} \nabla_\rho R_{\alpha\beta\gamma\delta} + R^\rho_{\alpha\lambda\mu} \nabla_\nu R_{\rho\beta\gamma\delta} + R^\rho_{\beta\lambda\mu} \nabla_\nu R_{\alpha\rho\gamma\delta} + R^\rho_{\gamma\lambda\mu} \nabla_\nu R_{\alpha\beta\rho\delta} + R^\rho_{\delta\lambda\mu} \nabla_\nu R_{\alpha\beta\gamma\rho} = 0 \quad (4)$$

so that if at any point  $p \in \mathcal{V}$  the matrix  $(R^{\alpha\beta}_{\gamma\delta})|_p$  of the Riemann tensor, considered as an endomorphism on the space of 2-forms  $\Lambda_2(p)$ , is non-singular, then we can multiply (4) by the inverse matrix of  $(R^{\alpha\beta}_{\gamma\delta})|_p$  getting at  $p$

$$\delta^\nu_{[\lambda} \nabla_{\mu]} R^{\alpha\beta\gamma\delta} + \delta^\alpha_{[\lambda} \nabla^\nu R_{\mu]}^{\beta\gamma\delta} + \delta^\beta_{[\lambda} \nabla^\nu R_{\mu]}^{\alpha\gamma\delta} + \delta^\gamma_{[\lambda} \nabla^\nu R_{\mu]}^{\alpha\beta\delta} + \delta^\delta_{[\lambda} \nabla^\nu R_{\mu]}^{\alpha\beta\gamma} = 0$$

which after contracting  $\nu$  with  $\lambda$  leads, using the second Bianchi identity, to

$$(n+1) \nabla_\mu R_{\alpha\beta\gamma\delta} = g_{\alpha\mu} \nabla_\rho R^\rho_{\beta\gamma\delta} + g_{\beta\mu} \nabla_\rho R_{\alpha}^{\rho}{}_{\gamma\delta} + g_{\gamma\mu} \nabla_\rho R_{\alpha\beta}^{\rho}{}_{\delta} + g_{\delta\mu} \nabla_\rho R_{\alpha\beta\gamma}^{\rho} . \quad (5)$$

Contracting here with  $g^{\mu\alpha}$  we easily get  $\nabla_\rho R^\rho{}_{\beta\gamma\delta} = 0$  and introducing this result into (5) we finally arrive at  $\nabla_\mu R_{\alpha\beta\gamma\delta} = 0$ . Thus, 2-symmetry (or 3-symmetry) implies local symmetry on a neighbourhood of any  $p \in \mathcal{V}$  at which the Riemann tensor matrix  $(R^{\alpha\beta}{}_{\gamma\delta})|_p$  is non-singular. This is the meaning of the word generic used above.

Keeping the meaning of the word “generic” in mind, this type of reasoning can be extended to arbitrary tensor fields—with possible stronger results depending on their order and symmetries—. For instance, one can prove the following general result.

**Proposition 2.1** *Let  $\tilde{T}$  be any tensor field. Around generic points, the vanishing of its second covariant derivative implies the vanishing of its first covariant derivative.*

*Proof.* Let us denote by  $T_{\alpha_1 \dots \alpha_q}$  the totally covariant tensor field equivalent to  $\tilde{T}$  by lowering all contravariant indices. Assume that  $\nabla_\lambda \nabla_\mu T_{\alpha_1 \dots \alpha_q} = 0$ , then the Ricci identity applied to  $\nabla_{[\lambda} \nabla_{\mu]} T_{\alpha_1 \dots \alpha_q} = 0$  and to  $\nabla_{[\lambda} \nabla_{\mu]} \nabla_\nu T_{\alpha_1 \dots \alpha_q} = 0$  provides, respectively

$$\sum_{i=1}^q R^\rho{}_{\alpha_i \lambda \mu} T_{\alpha_1 \dots \alpha_{i-1} \rho \alpha_{i+1} \dots \alpha_q} = 0, \quad (6)$$

$$R^\rho{}_{\nu \lambda \mu} \nabla_\rho T_{\alpha_1 \dots \alpha_q} + \sum_{i=1}^q R^\rho{}_{\alpha_i \lambda \mu} \nabla_\nu T_{\alpha_1 \dots \alpha_{i-1} \rho \alpha_{i+1} \dots \alpha_q} = 0. \quad (7)$$

At any  $p$  where  $\det(R^{\alpha\beta}{}_{\gamma\delta}) \neq 0$  then we have

$$\begin{aligned} \sum_{i=1}^q (g_{\alpha_i \lambda} T_{\alpha_1 \dots \alpha_{i-1} \mu \alpha_{i+1} \dots \alpha_q} - g_{\alpha_i \mu} T_{\alpha_1 \dots \alpha_{i-1} \lambda \alpha_{i+1} \dots \alpha_q}) &= 0, \\ g_{\nu \lambda} \nabla_\mu T_{\alpha_1 \dots \alpha_q} - g_{\nu \mu} \nabla_\lambda T_{\alpha_1 \dots \alpha_q} + \sum_{i=1}^q (g_{\alpha_i \lambda} \nabla_\nu T_{\alpha_1 \dots \alpha_{i-1} \mu \alpha_{i+1} \dots \alpha_q} - g_{\alpha_i \mu} \nabla_\nu T_{\alpha_1 \dots \alpha_{i-1} \lambda \alpha_{i+1} \dots \alpha_q}) &= 0 \end{aligned}$$

so that covariantly differentiating the first and subtracting the second one gets

$$g_{\nu \lambda} \nabla_\mu T_{\alpha_1 \dots \alpha_q} - g_{\nu \mu} \nabla_\lambda T_{\alpha_1 \dots \alpha_q} = 0$$

and contracting here  $\nu$  and  $\lambda$  one finally proves  $\nabla_\mu T_{\alpha_1 \dots \alpha_q} = 0$ . ■

**Corollary 2.1** *For any tensor field  $T$ , and at generic points, one has*

$$\overbrace{\nabla \dots \nabla}^k T = 0 \iff \nabla T = 0$$

for any  $k \geq 1$ . ■

These results apply in particular to the Riemann tensor and, actually, stronger results can be proven sometimes. As an interesting example, let us mention that by application of the previous results one can prove a conjecture in [26], namely, that all  $k$ -symmetric (and also all  $k$ -recurrent) spaces are necessarily of constant curvature on a neighbourhood of any  $p \in \mathcal{V}$  at which the Riemann tensor matrix is non-singular. As a matter of fact, let us prove the following slightly more general result

**Theorem 2.1** *All semi-symmetric spaces are of constant curvature at generic points.*

*Proof.* Assume that  $\nabla_{[\lambda} \nabla_{\mu]} R^\alpha{}_{\beta\gamma\delta} = 0$  and apply here the Ricci identity to get

$$R^\rho{}_{\alpha \lambda \mu} R_{\rho \beta \gamma \delta} + R^\rho{}_{\beta \lambda \mu} R_{\alpha \rho \gamma \delta} + R^\rho{}_{\gamma \lambda \mu} R_{\alpha \beta \rho \delta} + R^\rho{}_{\delta \lambda \mu} R_{\alpha \beta \gamma \rho} = 0 \quad (8)$$

so that as before, at any  $p$  with a non-singular Riemann tensor matrix,

$$\delta_{[\lambda}^{\alpha} R_{\mu]}^{\beta\gamma\delta} + \delta_{[\lambda}^{\beta} R^{\alpha}{}_{\mu]}{}^{\gamma\delta} + \delta_{[\lambda}^{\gamma} R^{\alpha\beta}{}_{\mu]}{}^{\delta} + \delta_{[\lambda}^{\delta} R^{\alpha\beta\gamma}{}_{\mu]} = 0$$

and contracting here  $\alpha$  with  $\lambda$  one derives  $(n-1)R_{\mu\beta\gamma\delta} = 2g_{\mu[\gamma}R_{\delta]\beta}$  immediately implying  $g_{\mu[\gamma}R_{\delta]\beta} = R_{\mu[\gamma}g_{\delta]\beta}$  from where contraction of  $\mu$  and  $\gamma$  gives  $(n-1)R_{\beta\gamma} = Rg_{\beta\gamma}$  which together with the previous formula for the Riemann tensor proves the result.  $\blacksquare$

In the previous proof, and in the rest of the paper,  $R_{\mu\nu} \equiv R^{\rho}{}_{\mu\rho\nu}$  and  $R \equiv g^{\mu\nu}R_{\mu\nu}$  denote the Ricci tensor and the scalar curvature, respectively.

Therefore, there is little room for spaces (necessarily of non-Euclidean signature) which are  $k$ -symmetric but *not* symmetric nor of constant curvature. It is remarkable that there have been many studies on 2-recurrent (or conformally 2-recurrent, see subsection 2.4) spaces, which certainly include the 2-symmetric ones, but surprisingly enough the assumption that they were *not* 2-symmetric was always, either implicitly or explicitly [68, 82, 83, 84, 54, 36, 37], made. Thus, the present paper tries to fill in this gap: the main purpose is to find these special manifolds for the case of 2-symmetry and Lorentzian signature. This will be done from section 3 onwards. Before that, let us collect some general results valid in arbitrary signature and some possible nomenclature concerning the conformal and Ricci curvature tensors.

## 2.2. Identities in 2-symmetric semi-Riemannian manifolds

Some basic tensor calculation is needed, mainly to prove quadratic identities which hold in general 2-symmetric spaces of any signature and that, with the help of the lemmas in section 3, can be used to get the sought results.

To start with, we need a generalization to non-generic points —i.e. to the case with a possibly degenerate Riemann tensor matrix— of the calculations and results presented in subsection 2.1.

**Lemma 2.1** *Let  $(\mathcal{V}, g)$  be an  $n$ -dimensional 2-symmetric semi-Riemannian manifold of any signature. If  $\nabla_{\lambda}\nabla_{\mu}T_{\mu_1\dots\mu_q} = 0$  then*

$$\sum_{i=1}^q \nabla_{\nu} R^{\rho}{}_{\alpha_i\lambda\mu} T_{\alpha_1\dots\alpha_{i-1}\rho\alpha_{i+1}\dots\alpha_q} - R^{\rho}{}_{\nu\lambda\mu} \nabla_{\rho} T_{\alpha_1\dots\alpha_q} = 0, \quad (9)$$

$$(\nabla_{\nu} R^{\rho}{}_{\tau\lambda\mu} + \nabla_{\tau} R^{\rho}{}_{\nu\lambda\mu}) \nabla_{\rho} T_{\mu_1\dots\mu_q} = 0, \quad (10)$$

$$(\nabla_{\nu} R^{\rho}{}_{\mu} - \nabla_{\mu} R^{\rho}{}_{\nu}) \nabla_{\rho} T_{\mu_1\dots\mu_q} = 0, \quad (\nabla^{\rho} R_{\mu\nu} - 2\nabla_{\nu} R^{\rho}{}_{\mu}) \nabla_{\rho} T_{\mu_1\dots\mu_q} = 0. \quad (11)$$

*Proof.* As (6) and (7) hold, by covariantly differentiating the first and subtracting the second one gets (9). Computing the covariant derivative of this equation and that of (7), using the 2-symmetry, and subtracting them one arrives immediately at (10). Contracting here  $\tau, \nu$ , or  $\tau, \lambda$ , one gets the two in (11), respectively.  $\blacksquare$

Recall the decomposition of the Riemann tensor,

$$R_{\alpha\beta\lambda\mu} = C_{\alpha\beta\lambda\mu} + \frac{2}{n-2} (R_{\alpha[\lambda}g_{\mu]\beta} - R_{\beta[\lambda}g_{\mu]\alpha}) - \frac{R}{(n-1)(n-2)} (g_{\alpha\lambda}g_{\beta\mu} - g_{\alpha\mu}g_{\beta\lambda}) \quad (12)$$

where  $C_{\alpha\beta\lambda\mu}$  is the conformal or Weyl curvature tensor, which satisfies the same symmetry properties as the Riemann tensor and is also traceless

$$C_{\alpha\beta\lambda\mu} = C_{[\alpha\beta][\lambda\mu]}, \quad C_{\alpha[\beta\lambda\mu]} = 0, \quad C^{\rho}{}_{\beta\rho\mu} = 0$$

where of course the first two imply that  $C_{\alpha\beta\lambda\mu} = C_{\lambda\mu\alpha\beta}$ . Then the following general identities hold for arbitrary semi-symmetric spaces

**Lemma 2.2** *The Riemann, Ricci and Weyl tensors of an  $n$ -dimensional semi-symmetric semi-Riemannian manifold of any signature satisfy (8) as well as*

$$R_{\rho(\mu}R^{\rho}_{\nu)\alpha\beta} = 0, \quad R^{\rho}_{\mu[\alpha\beta}R_{\gamma]\rho} = 0, \quad C^{\rho}_{\mu[\alpha\beta}R_{\gamma]\rho} = 0, \quad R^{\rho\sigma}R_{\rho\mu\sigma\nu} = R_{\mu}{}^{\rho}R_{\rho\nu}, \quad (13)$$

$$R^{\rho}_{\alpha\lambda\mu}C_{\rho\beta\gamma\delta} + R^{\rho}_{\beta\lambda\mu}C_{\alpha\rho\gamma\delta} + R^{\rho}_{\gamma\lambda\mu}C_{\alpha\beta\rho\delta} + R^{\rho}_{\delta\lambda\mu}C_{\alpha\beta\gamma\rho} = 0, \quad (14)$$

$$(n-2)(C_{\rho[\alpha}{}^{\lambda\mu}C^{\rho}_{\beta]\gamma\delta} + C_{\rho[\gamma}{}^{\lambda\mu}C^{\rho}_{\delta]\alpha\beta}) - 2\left(R_{[\alpha}{}^{[\lambda}C^{\mu]}_{\beta]\gamma\delta} + R_{[\gamma}{}^{[\lambda}C^{\mu]}_{\delta]\alpha\beta}\right) - 2\left(R_{\rho}{}^{[\lambda}\delta^{\mu]}_{[\alpha}C^{\rho}_{\beta]\gamma\delta} + R_{\rho}{}^{[\lambda}\delta^{\mu]}_{[\gamma}C^{\rho}_{\delta]\alpha\beta}\right) + 2\frac{R}{n-1}\left(\delta_{[\alpha}^{[\lambda}C^{\mu]}_{\beta]\gamma\delta} + \delta_{[\gamma}^{[\lambda}C^{\mu]}_{\delta]\alpha\beta}\right) = 0 \quad (15)$$

and their non-written traces, such as the appropriate specializations of (11).

*Proof.* The first in (13) is the trace of (8) (or the Ricci identity for the Ricci tensor), the fourth is its trace, the second follows from the first by taking a cyclic permutation of indices, and the third follows from the second by using (12). On the other hand, (14) is the Ricci identity for the Weyl tensor and finally (15) follows from (14) on using (12) again. ■

In the particular case of 2-symmetric manifolds we also have that

**Lemma 2.3** *The Riemann, Ricci and Weyl tensors of an  $n$ -dimensional 2-symmetric semi-Riemannian manifold of any signature satisfy (4) as well as*

$$\nabla_{(\tau}R^{\rho}_{\nu)\lambda\mu}\nabla_{\rho}R_{\alpha\beta\gamma\delta} = 0, \quad \nabla_{(\tau}R^{\rho}_{\nu)\lambda\mu}\nabla_{\rho}C_{\alpha\beta\gamma\delta} = 0, \quad \nabla_{(\tau}R^{\rho}_{\nu)\lambda\mu}\nabla_{\rho}R_{\alpha\beta} = 0, \quad (16)$$

and their non-written traces, such as the appropriate specializations of (11).

*Proof.* These are the particularization of (10) to the various curvature tensors. ■

### 2.3. Special vector fields

We briefly collect an important result, and its consequences, that has been used classically (see e.g. [81, 56]) to study the second order symmetric semi-Riemannian manifolds. This will be useful later in the proof of Proposition 3.1. The idea is to explore the consequences of the existence of a vector field whose covariant derivative is proportional to the metric, so that it has vanishing second covariant derivative.

**Lemma 2.4** *Let  $(V, g)$  be 2-symmetric with arbitrary signature. If a 1-form  $v_{\mu}$  satisfies*

$$\nabla_{\nu}v_{\mu} = c g_{\mu\nu}$$

for a constant  $c$ , then either  $c = 0$  or the manifold is locally symmetric.

*Proof.* The condition of the lemma implies that  $v^{\mu}$  is a homothetic vector [88, 75]:

$$(\mathcal{L}_{\vec{v}}g)_{\mu\nu} = 2cg_{\mu\nu}$$

where  $\mathcal{L}_{\vec{v}}$  denotes the Lie derivative along  $\vec{v}$ . From standard results [88]  $(\mathcal{L}_{\vec{v}}R)^{\alpha}_{\beta\gamma\delta} = 0$ , and as the covariant derivative commutes with the Lie derivative along homothetic vectors, then  $(\mathcal{L}_{\vec{v}}\nabla R)_{\mu}{}^{\alpha}_{\beta\gamma\delta} = 0$ . Expanding this expression and using on the one hand the 2-symmetry and on the other that  $\nabla_{\nu}v^{\mu} = c\delta^{\mu}_{\nu}$  one derives

$$(\mathcal{L}_{\vec{v}}\nabla R)_{\mu}{}^{\alpha}_{\beta\gamma\delta} = 3c\nabla_{\mu}R^{\alpha}_{\beta\gamma\delta} = 0$$

from where either  $c = 0$  or the space is locally symmetric. ■

#### 2.4. Conformal and Ricci $k$ -symmetry

All the previous definitions of locally higher-order symmetric spaces can be straightforwardly generalized by substituting the Weyl tensor  $C^\alpha_{\beta\gamma\delta}$  or the Ricci tensor  $R_{\mu\nu}$  for the Riemann tensor in the defining conditions. Thus, one uses the terms *locally conformally symmetric* [19, 1] if

$$\nabla_\mu C^\alpha_{\beta\gamma\delta} = 0,$$

and *locally Ricci-symmetric* [76] if

$$\nabla_\rho R_{\mu\nu} = 0.$$

Similarly for conformally (or Ricci) recurrent [2, 59, 67, 57], conformally (or Ricci) 2-recurrent [20, 55, 68], and  $k$ -recurrent. We can also adopt such a convention and use the terms conformal  $k$ -symmetric and Ricci  $k$ -symmetric in the obvious way.

Of course,  $k$ -symmetry implies both conformal  $k$ -symmetry and Ricci  $k$ -symmetry, but any of the latter does not by itself imply the former, see e.g. [69]. It must be noted that, as above,

- proper Riemannian Ricci  $k$ -symmetric spaces are Ricci-symmetric,
- proper Riemannian conformally  $k$ -symmetric manifolds are conformally symmetric [81], and furthermore either locally symmetric or conformally flat [29].

The interplay between the different mentioned symmetry and recurrent conditions of any order  $k$  has been studied in many papers, e.g [83, 42, 81, 69, 70, 30, 31, 45, 36, 37]. One can also check the Appendix in [28] for an exhaustive list of curvature conditions and their overlappings.

### 3. Lorentzian 2-symmetry

There are several ways to attack the problem of  $k$ -symmetric and  $k$ -recurrent spaces, among them I would like to cite the following:

- (i) pure classical standard tensor calculus by using the Ricci and Bianchi identities. This allows us to obtain necessary restrictions such as those presented already in subsection 2.2;
- (ii) study of covariantly constant (also called *parallel*) tensor and vector fields, and their implications on the manifold local holonomy structure. This will be dealt with mainly, but not only, in subsection 3.1;
- (iii) probing the existence of Killing or homothetic vector fields and consequences thereof (subsection 2.3 and proof of proposition 3.1);
- (iv) implications on the curvature invariants and in particular their possible constancy or vanishing. This will be largely analyzed in subsection 3.2.

Of course, all these methods are clearly interrelated. Here, I shall follow a mixed strategy, using results from all of them, trying to optimize the path to the sought for results. It turns out that the so-called “superenergy” and/or causal-tensor techniques [73, 6] are extremely useful for getting the results as they provide properties and positive quantities associated with tensors and their tensor products that can be used to, at least partly, replace the ordinary positive-definite metric available in proper Riemannian cases.

#### 3.1. Holonomy and reducibility in Lorentzian manifolds

To start with, we will need some basic lemmas on local holonomy structure. The classical result here is the de Rham decomposition theorem [66, 51] for positive-definite metrics. However, this theorem does not hold as such for other signatures, and one has to introduce the so-called *non-degenerate reducibility* due



to Wu [85, 86, 87], who extended de Rham's results to indefinite metrics. See also [4] for the particular case of Lorentzian signature.

To fix the ideas, recall that the holonomy group [51] of  $(\mathcal{V}, g)$  is called

- *reducible* (when acting on the tangent spaces) if it leaves a non-trivial subspace of  $T_p\mathcal{V}$  invariant;
- *non-degenerately reducible* if it leaves a non-degenerate subspace—that is, such that the restriction of the metric is non-degenerate—invariant.

Fortunately, we will only need a simple result which relates the existence of covariantly constant tensors to the holonomy group of the manifold in the case of Lorentzian signature. This is a synthesis (adapted to our purposes) of the results in [46] (see also [63, 75]) but generalized to arbitrary dimension  $n$ :  $\Sigma$

**Lemma 3.1** *Let  $D \subset \mathcal{V}$  be a simply connected domain of an  $n$ -dimensional Lorentzian manifold  $(\mathcal{V}, g)$  and assume that there exists a non-zero covariantly constant symmetric tensor field  $h_{\mu\nu}$  not proportional to the metric. Then  $(D, g)$  is reducible, and further it is not non-degenerately reducible only if there exists a null covariantly constant vector field which is the unique (up to a constant of proportionality) constant vector field.*

**Remarks:**

- If there is a covariantly constant 1-form  $v_\mu$ , then so is obviously  $h_{\mu\nu} = v_\mu v_\nu$  and the manifold (arbitrary signature) is reducible, the Span of  $v^\mu$  being invariant by the holonomy group. If  $v_\mu$  is *not* null, then  $(\mathcal{V}, g)$  is actually *non-degenerately* reducible (also called decomposable, see [63, 75]). In this case, the metric can be decomposed into two orthogonal parts as  $g_{\mu\nu} = c v_\mu v_\nu + (g_{\mu\nu} - c v_\mu v_\nu)$ , where  $c = 1/(v^\mu v_\mu)$  is constant. Thus, necessarily  $g_{\mu\nu}$  is a *flat extension* [71] of a  $(n-1)$ -dimensional non-degenerate metric  $g_{\mu\nu} - c v_\mu v_\nu$ .
- If there is a covariantly constant non-symmetric tensor  $H_{\mu\nu}$ , then its symmetric part is also a constant tensor at our disposal, so that one can put  $h_{\mu\nu} = H_{(\mu\nu)}$  in the lemma unless this vanishes, see next remark.
- In the case that  $H_{\mu\nu} = H_{[\mu\nu]} \neq 0$  is antisymmetric, then in fact one can define  $H_{\mu\rho} H_\nu{}^\rho = h_{\mu\nu}$ , which is symmetric, covariantly constant, non-zero and *not* proportional to the metric if  $n > 2$ . These last two statements follow for example from lemma 3.2 and corollary 4.1 in [6].
- Actually, the previous point can be generalized to an arbitrary covariantly constant  $p$ -form  $\Sigma_{\mu_1 \dots \mu_p}$  (with  $p < n$ ) by defining  $h_{\mu\nu} = \Sigma_{\mu\rho_2 \dots \rho_p} \Sigma_\nu{}^{\rho_2 \dots \rho_p}$ .

*Proof.* The proof relies on the algebraic (or Segre) classification of 2-index symmetric covariant tensors, according to its possible eigenvalues and eigenvectors *with respect to the metric*, that is, the solution to the problem  $(T_{\mu\nu} - \lambda g_{\mu\nu})b^\nu = 0$ . Using the Segre notation, and as is known [58] pp.261-262, only four different types occur, namely  $[1, 1 \dots 1]$ ,  $[21 \dots 1]$ ,  $[31 \dots 1]$  and  $[z\bar{z}1 \dots 1]$ , where a comma separates the eigenvalue associated to the timelike eigenvector in case this exists, and the notation  $z\bar{z}$  is used to indicate a pair of complex conjugate eigenvalues. Degeneracies are denoted, as usual, by round brackets, so for example  $[(1, 1 \dots 1)]$  is the Segre type of the metric tensor.

Following [46], one can first of all show that the Segre type of  $h_{\mu\nu}$  is the same everywhere on  $D$  and that its eigenvalues are constant on  $D$ . This follows by computing the eigenvalues and eigenvectors at a neighbourhood  $U(p) \subset D$ , which can be taken as differentiable on adequate  $U(p)$ . Then, for an arbitrary curve on  $U(p)$  with tangent vector  $\vec{w}$  one can parallelly transport the eigenvectors to get

$$h_{\mu\nu} e^\nu = \lambda e_\mu \implies \nabla_{\vec{w}}(h_{\mu\nu} e^\nu) = 0 = (\nabla_{\vec{w}} \lambda) e_\mu$$

$\Sigma$  This result also follows from corollary 4 and proposition 5 in [33], and can be seen as a consequence of the results on the holonomy group of irreducible Lorentzian manifolds found in [34].

and given that  $\vec{w}$  is arbitrary one readily gets  $\lambda = \text{const.}$  on  $D$ .

Suppose then that there is an eigenvalue  $\lambda$  corresponding to a spacelike eigenvector (which is always the case if  $n > 3$ ). Suppose further that  $\lambda$  is not the eigenvalue of the timelike eigenvector for case  $[1, 1 \dots 1]$ , or of the null eigenvector for the cases  $[21 \dots 1]$  or  $[31 \dots 1]$ . In other words, by relabelling the spacelike eigenvalue if necessary, we are considering all cases other than  $[(1, 1 \dots 1)]$ ,  $[(21 \dots 1)]$  and  $[(31 \dots 1)]$ . Then, there is an orthonormal basis of eigenvectors  $\{\vec{e}_A\}$  for the *spacelike* eigenspace corresponding to  $\lambda$ , where  $A, B \dots = 1, \dots, q$  and  $q$  is the dimension of the eigenspace, so that using that  $\lambda$  is constant on  $D$

$$h^\mu{}_\nu e_A^\nu = \lambda e_A^\mu \implies h^\mu{}_\nu \nabla_\rho e_A^\nu = \lambda \nabla_\rho e_A^\mu \implies \nabla_\rho e_A^\nu = \sum_B Q_{\rho AB} e_B^\nu$$

for some one-forms  $Q_{\rho AB}$ . However, from the orthonormal property one easily gets  $Q_{\rho AB} = -Q_{\rho BA}$ , so that an elementary calculation provides

$$\nabla_\rho \left( \sum_A e_A^\mu e_A^\nu \right) = \sum_{AB} Q_{\rho AB} (e_A^\mu e_B^\nu + e_B^\mu e_A^\nu) = 0$$

which tells us that the tensor  $k_{\mu\nu} = \sum_A e_{A\mu} e_{A\nu}$  is covariantly constant and also, being idempotent ( $k_{\mu\rho} k^\rho{}_\nu = k_{\mu\nu}$ ), a projector. It follows that the spacetime is non-degenerately reducible and the metric can be non-degenerately decomposed *at least* as  $g_{\mu\nu} = k_{\mu\nu} + (g_{\mu\nu} - k_{\mu\nu})$ .

Given that the tensor  $h_{\mu\nu}$  is not proportional to the metric, the only remaining cases for its Segre type are thus  $[(21 \dots 1)]$  and  $[(31 \dots 1)]$ . Then, a reasoning similar to that in [46], as the extra spatial dimensions are immaterial here (but the Lorentzian signature is essential), shows that the unique null eigendirection can be rescaled to provide a null covariantly constant vector field. This must be the only parallel vector field or otherwise the previous Remark i would imply that the spacetime is a flat extension (ergo non-degenerately reducible) of a  $(n - 1)$ -dimensional non-degenerate manifold. (Actually, in the case  $[(31 \dots 1)]$  there are spacelike parallel vector fields). ■

**Remark:** Of course, similar reasonings can be applied to the different non-degenerate eigenspaces of  $h_{\mu\nu}$ , providing a more detailed decomposition of the spacetime. One should bear in mind this for the rest of the paper together with the basic decomposition  $g_{\mu\nu} = k_{\mu\nu} + (g_{\mu\nu} - k_{\mu\nu})$  associated to each non-degenerate eigenspace.

### 3.2. Curvature concomitants in 2-symmetric Lorentzian manifolds

Recall that a curvature scalar invariant, see e.g. [75, 40, 65, 63] and references therein, is a scalar constructed polynomially from the Riemann tensor, the metric, the covariant derivative and possibly the volume element  $n$ -form of  $(\mathcal{V}, g)$ . They are fundamental in the sense that they are scalars which depend only on the metric, and not on the basis used to compute them. They have been largely studied in General Relativity ( $n = 4$ ) as they provide algebraic classification of spacetimes and important physical information, see [63, 75, 65] and references therein. More generally, one can define curvature 1-form and curvature rank- $r$  tensorial “invariants” by using the same ingredients and in the same manner but leaving  $1, \dots, r$  free indices [40], respectively. These tensorial quantities are sometimes called “curvature concomitants” (see e.g. [72] p.15 and p.164), other times “curvature covariants” (e.g. [62] p.260.) Here we shall use the former.

One can thus associate three fundamental natural numbers to all curvature tensorial concomitants, called the *rank*, the *degree*, and the *order*:

- the rank is the tensor rank of the tensorial concomitant —that is, the number  $r$  of free indices—, so that  $r = 0$  for scalar invariants.

- the degree is the (maximum) power of the Riemann tensor used in the tensorial concomitant. Hence, they are called linear, quadratic, cubic, etcetera if they are linear, quadratic, cubic, and so on, on the Riemann tensor.
- finally, the order is the (maximum) number of covariant derivatives involved in one single summand of the tensorial concomitant.

Curvature tensorial concomitants are said to be homogeneous with respect to the order if they have the same number of covariant derivatives in all its terms, and similarly for homogeneous with respect to the degree. Of course, all non-homogeneous tensorial concomitants can be broken into their respective homogeneous pieces, and therefore in what follows we will only consider the homogeneous ones.

The only linear scalar invariant is  $R$  (order zero), and examples of quadratic concomitants are  $R_{\mu\nu}R^{\mu\nu}$ ,  $C_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$  (order zero),  $\nabla_\mu R_{\alpha\beta\gamma\delta}\nabla^\mu R^{\alpha\beta\gamma\delta}$  (order 2),  $R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} + \nabla_\rho R_{\mu\nu}\nabla^\rho R^{\mu\nu}$  (non-homogeneous),  $R\nabla_\mu R$  (order 1, rank 1),  $\nabla^\rho C^{\alpha\beta\mu\nu}\nabla_\rho R_{\beta\nu}$  (order 2, rank 2).

The following are very useful lemmas.

**Lemma 3.2** *Let  $D \subset \mathcal{V}$  be a simply connected domain and let  $g$  have arbitrary signature. Any curvature 1-form concomitant which is covariantly constant must be necessarily null (possibly zero).*

*Proof.* If it were non-null, due to the Remark i of Lemma 3.1,  $(D, g)$  would be a flat extension of a  $(n-1)$  space, but this is clearly impossible as the curvature would then be that of only the reduced  $(n-1)$  part. ■

**Lemma 3.3** *Let  $(D, g)$  be as in lemma 3.1. Any curvature 2-form ( $2 < n$ ) concomitant which is covariantly constant is null (possibly zero), and if non-zero there is a null covariantly constant vector field.*

(For the definition of null 2-form, see e.g. [75, 6])

*Proof.* From the Remark iii of Lemma 3.1, and using the notation there,  $(D, g)$  is reducible. If the 2-form is null and non-zero, then in fact the parallel tensor  $h_{\mu\nu} = k_\mu k_\nu$  where  $k_\mu$  defines the unique null direction determined by the null 2-form. Therefore in this case  $k_\mu$  must be covariantly constant. If the 2-form were non-null, then  $(D, g)$  would be non-degenerately reducible so that the 2-form concomitant itself decomposes as the sum of the respective reduced 2-forms, some of which may be zero. The non-zero ones would necessarily be such that the corresponding reduced  $h_{\mu\nu}$  is proportional to the metric of a 2-dimensional subspace defined by the decomposition. As this 2-space is necessarily 2-symmetric, by standard results it must in fact be of constant curvature and thus the curvature 2-form concomitant would have to vanish anyway. ■

A very simple yet useful lemma is the following.

**Lemma 3.4** *Let  $(\mathcal{V}, g)$  be an  $n$ -dimensional 2-symmetric Lorentzian manifold. Then*

- *all non-zero curvature tensorial concomitants have a degree necessarily greater or equal than the order;*
- *all curvature tensorial concomitants with order equal to degree are necessarily parallel.*

*Proof.* As the spacetime is 2-symmetric, there will be no non-zero curvature tensorial concomitants involving derivatives of the Riemann tensor of order higher than one. This proves the first assertion and, mutatis mutandis, the second too. ■

With these lemmas in hand, one of the main results can be proven. Observe that from lemma 3.2 it follows that, in 2-symmetric spaces, either  $R$  is constant or  $\nabla_\mu R$  is null and covariantly constant. This is a particular example of the following general result.

**Proposition 3.1** *Let  $D \subset \mathcal{V}$  be a simply connected domain of an  $n$ -dimensional 2-symmetric Lorentzian manifold  $(\mathcal{V}, g)$ . Then either*

- *all scalar invariants of the Riemann tensor of order  $m$  and degree up to  $m+2$  are constant on  $D$ ; or*

- there is a covariantly constant null vector field on  $D$ .

*Proof.* Let  $I$  be a scalar invariant of order  $m$ . If its degree is also  $m$  then obviously (see Lemma 3.4)  $\nabla I = 0$  so that  $I$  is constant. If its degree is  $m + 1$ , then  $\nabla_\nu \nabla_\mu I = 0$  from where  $v_\mu = \nabla_\mu I$  is a covariantly constant curvature 1-form concomitant, so that from Lemma 3.2 it must be null or zero, in the last case implying that  $I$  is constant. Finally, if the degree is  $m + 2$ , then  $\nabla_\rho \nabla_\nu \nabla_\mu I = 0$  so that  $h_{\mu\nu} = \nabla_\nu \nabla_\mu I = \nabla_\nu v_\mu$  is a covariantly constant symmetric tensor. According to Lemma 3.1, there arise three possibilities: there is a null covariantly constant vector field, or  $(D, g)$  is non-degenerately reducible and there is no null parallel vector field, or  $h_{\mu\nu} = cg_{\mu\nu}$  for a constant  $c$ . In the first possibility we are done; in the second case, the space is decomposed into several irreducible and necessarily 2-symmetric parts, in particular one for which  $h_{\mu\nu}$  is proportional to its reduced metric tensor. As  $I$  will be a function of the corresponding scalar invariants of the irreducible parts, everything reduces effectively to only the third possibility. But then Lemma 2.4 implies that either  $c = 0$  or the (reduced) space is locally symmetric. In both cases  $h_{\mu\nu} = \nabla_\nu v_\mu = 0$  implying that  $v_\mu = \nabla_\mu I$  must be covariantly constant. Thus, the same reasoning as before using Lemma 3.2 ends the proof. ■

There are, of course, ways to obtain particular results of this lemma in a direct manner. For example, by contracting  $\alpha$  and  $\gamma$ ,  $\beta$  and  $\delta$ ,  $\nu$  and  $\mu$  in (4) one gets  $R^\rho{}_\mu \nabla_\rho R = 0$ , so that contracting here with  $\nabla^\mu$  and using the contracted Bianchi identity

$$\nabla_\mu R^\mu{}_\nu = \frac{1}{2} \nabla_\nu R \quad (17)$$

the condition  $\nabla_\nu R \nabla^\nu R = 0$  is obtained. Then either  $R = \text{const.}$  or  $\nabla_\nu R$  is covariantly constant and null. Nevertheless, there are other results which are much more difficult to prove using pure tensor calculus. For instance,

$$\nabla_\alpha (R^{\mu\nu} R_{\mu\nu}) = 0 = R^{\mu\nu} \nabla_\alpha R_{\mu\nu} \quad (18)$$

which is obviously contained in Proposition 3.1 if there is no null parallel vector field.

The previous proposition has immediate consequences providing more information about curvature tensorial concomitants. For instance

**Corollary 3.1** *Under the conditions of Proposition 3.1, either there is a covariantly constant null vector field on  $D$  or the following statements hold*

- (i) *All curvature scalar invariants of any order and degree formed as functions of the homogeneous ones of order  $m$  and degree up to  $m + 2$  are constant on  $D$ ;*
- (ii) *All curvature 1-form concomitants of order  $m$  and degree up to  $m + 1$  are zero.*
- (iii) *All curvature scalar invariants with order equal to degree vanish.*
- (iv) *All curvature rank-2 tensorial concomitants with order equal to degree are zero.*

**Remark:** Of course, it can happen that the mentioned curvature concomitants vanish *and* there is a covariantly constant null vector field too.

*Proof.* The first statement is trivial. For the second, choose any 1-form concomitant  $I_\mu$  of order  $m$ . If the degree is also  $m$ , then obviously (Lemma 3.4)  $I_\mu$  is covariantly constant so that Lemma 3.2 applies. If its degree is  $m + 1$ , then its exterior differential  $F_{\mu\nu} = \nabla_{[\nu} I_{\mu]}$  is a covariantly constant 2-form and thus Lemma 3.3 implies that, if there is no null parallel vector field (which means that  $F_{\mu\nu}$  cannot be non-zero and null),  $\nabla_{[\nu} I_{\mu]} = 0$ , so that  $I_\mu$  is an exact 1-form and therefore  $I_\mu$  is the differential of a function  $f$ ,  $I_\mu = \nabla_\mu f$ . If  $f$  happens to be a scalar invariant  $f = I$ , then  $I$  is of order  $m - 1$  and degree  $m + 1$ . As  $I$  is constant due to Proposition 3.1, then  $I_\mu = 0$ . But even in the case that  $f$  is not a scalar curvature invariant, the symmetric tensor  $h_{\mu\nu} = \nabla_\nu I_\mu = \nabla_\nu \nabla_\mu f$  is covariantly constant, and a reasoning totally analogous to that in the end of the proof of Proposition 3.1 implies that  $I_\mu$  is parallel, so that Lemma 3.2 applies again.

For the third statement, take any scalar invariant  $I$  with order and degree equal to  $m$ . Then obviously this is the divergence  $I = \nabla_\mu I^\mu$  of a 1-form concomitant  $I_\mu$  of order  $m-1$  and degree  $m$ . As  $I_\mu = 0$  due to point *ii*,  $I = 0$ .

Finally, for the point *iv*, as the curvature rank-2 tensorial concomitant  $I_{\mu\nu}$  has the order equal to the degree, it is covariantly constant  $\nabla_\rho I_{\mu\nu} = 0$ . Furthermore, due to point *iii*, its trace vanishes  $I^\nu{}_\nu = 0$ . Its symmetric part  $I_{(\mu\nu)}$  is therefore covariantly constant and *not* proportional to the metric. By using Lemma 3.1, following the same reasoning as in Proposition 3.1 one easily proves  $I_{(\mu\nu)} = 0$  unless there is a null parallel vector field. If this is not the case, then  $I_{\mu\nu} = I_{[\mu\nu]}$  is a covariantly constant 2-form and then lemma 3.3 implies finally that  $I_{\mu\nu} = 0$ .  $\blacksquare$

There is a very long list of vanishing curvature scalar and tensorial concomitants as a result of this proposition—if there is no null parallel vector field—. The following is the list of the quadratic ones (only an independent set [40], apart from (18), is given, also omitting those containing  $\nabla_\mu R = 0$ ):

$$R^{\mu\nu}\nabla_\mu R_{\nu\alpha} = 0, R^{\mu\nu\rho\alpha}\nabla_\mu R_{\nu\rho} = 0, R^{\mu\nu\rho\sigma}\nabla_\mu R_{\nu\rho\sigma\alpha} = 0 = R^{\mu\nu\rho\sigma}\nabla_\alpha R_{\mu\nu\rho\sigma}, \quad (19)$$

$$\nabla_\alpha R^{\mu\nu}\nabla_\beta R_{\mu\nu} = \nabla_\mu R_{\nu\beta}\nabla_\alpha R^{\mu\nu} = \nabla_\mu R_{\nu\alpha}\nabla^\mu R^\nu{}_\beta = \nabla_\mu R_{\nu\alpha}\nabla^\nu R^\mu{}_\beta = 0, \quad (20)$$

$$\nabla^\mu R^{\nu\rho}\nabla_\alpha R_{\beta\rho\mu\nu} = \nabla^\mu R^{\nu\rho}\nabla_\mu R_{\alpha\nu\beta\rho} = 0, \nabla_\alpha R^{\mu\nu\rho\sigma}\nabla_\beta R_{\mu\nu\rho\sigma} = \nabla^\sigma R^{\mu\nu\rho\alpha}\nabla_\sigma R_{\mu\nu\rho\beta} = 0 \quad (21)$$

where of course the traces of (20-21) vanish, and one can also write the same expressions using the Weyl tensor instead of the Riemann tensor.

#### 4. Main results

All necessary results to prove the main theorems have now been gathered. Let us start with an important result which will be derived by using the so-called causal tensors and “superenergy” techniques [73, 6]. The idea is that a tensor such as  $\nabla_\mu R_{\alpha\beta}$ , whose tensor square has all double traces vanishing—see (20)—, and from (11) also satisfies

$$(\nabla_\nu R^\rho{}_\mu - \nabla_\mu R^\rho{}_\nu)\nabla_\rho R_{\alpha\beta} = 0, \quad (\nabla^\rho R_{\mu\nu} - 2\nabla_\nu R^\rho{}_\mu)\nabla_\rho R_{\alpha\beta} = 0, \quad (22)$$

can only be non-zero if  $\nabla_\mu R_{\alpha\beta} = k_\mu(k_\alpha p_\beta + k_\beta p_\alpha)$  where  $k_\mu$  is a null vector field orthogonal to  $p_\mu$ . But as the tensor  $\nabla_\mu R_{\alpha\beta}$  is covariantly constant, so is the null vector field.

**Theorem 4.1** *Let  $D \subset \mathcal{V}$  be a simply connected domain of an  $n$ -dimensional 2-symmetric Lorentzian manifold  $(\mathcal{V}, g)$ . Then, if there is no null covariantly constant vector field on  $D$ ,  $(D, g)$  is the direct product of irreducible submanifolds each of which is either Ricci-flat ( $R_{\mu\nu} = 0$ ) or locally symmetric.*

*Proof.* The first part of the reasoning is algebraic and can be performed at any point  $p \in D$ . Take the double (2,1)-form  $\nabla_{[\alpha} R_{\beta]\lambda}$  and construct its basic superenergy tensor [73]:

$$\begin{aligned} T_{\alpha\beta\lambda\mu} = & (\nabla_\alpha R_{\rho\lambda} - \nabla_\rho R_{\alpha\lambda})(\nabla_\beta R^\rho{}_\mu - \nabla^\rho R_{\beta\mu}) + (\nabla_\alpha R_{\rho\mu} - \nabla_\rho R_{\alpha\mu})(\nabla_\beta R^\rho{}_\lambda - \nabla^\rho R_{\beta\lambda}) - \\ & - \frac{1}{2}g_{\alpha\beta}(\nabla_\sigma R_{\rho\lambda} - \nabla_\rho R_{\sigma\lambda})(\nabla^\sigma R^\rho{}_\mu - \nabla^\rho R^{\sigma\mu}) - g_{\lambda\mu}(\nabla_\alpha R_{\rho\sigma} - \nabla_\rho R_{\alpha\sigma})(\nabla_\beta R^{\rho\sigma} - \nabla^\rho R_{\beta}{}^\sigma) + \\ & + \frac{1}{4}g_{\alpha\beta}g_{\lambda\mu}(\nabla_\tau R_{\rho\sigma} - \nabla_\rho R_{\tau\sigma})(\nabla^\tau R^{\rho\sigma} - \nabla^\rho R^{\tau\sigma}). \end{aligned}$$

This is the (essentially unique) tensor which is quadratic on  $\nabla_{[\mu} R_{\alpha]\beta}$  and is “causal” or “future” [73, 6], that is to say, it satisfies the *dominant property*:

$$T_{\alpha\beta\lambda\mu}u_1^\alpha u_2^\beta u_3^\lambda u_4^\mu \geq 0$$

for every choice of future-pointing vectors  $\{u_1^\alpha, u_2^\beta, u_3^\lambda, u_4^\mu\}$  (as a matter of fact, this is strictly positive if all the previous vectors are timelike). It is known that the tensor  $T_{\alpha\beta\lambda\mu}$  vanishes if and only if so does

$\nabla_{[\mu} R_{\alpha]\beta}$ , and if and only if its contraction in all indices with a timelike vector vanishes. Another direct property is  $T_{\alpha\beta\lambda\mu} = T_{(\alpha\beta)(\lambda\mu)}$ . In our case, first of all by use of (20) the tensor becomes

$$T_{\alpha\beta\lambda\mu} = (\nabla_{\alpha} R_{\rho\lambda} - \nabla_{\rho} R_{\alpha\lambda})(\nabla_{\beta} R^{\rho}_{\mu} - \nabla^{\rho} R_{\beta\mu}) + (\nabla_{\alpha} R_{\rho\mu} - \nabla_{\rho} R_{\alpha\mu})(\nabla_{\beta} R^{\rho}_{\lambda} - \nabla^{\rho} R_{\beta\lambda})$$

and using repeatedly (22) this finally simplifies to

$$T_{\alpha\beta\lambda\mu} = \nabla_{\alpha} R_{\rho\lambda} \nabla_{\beta} R^{\rho}_{\mu} + \nabla_{\alpha} R_{\rho\mu} \nabla_{\beta} R^{\rho}_{\lambda}. \quad (23)$$

Now, one has to use several elementary properties of causal tensors. As  $T_{\alpha\beta\lambda\mu}$  is future (theorem 4.1 in [73]), its contraction in any number of indices with arbitrary future-pointing vectors is also a causal tensor (property 2.2 in [6]), so that in particular, for an arbitrary timelike vector  $u^{\mu}$ , one has that

$$T_{\alpha\beta\lambda\mu} u^{\alpha} u^{\beta} = 2 u^{\sigma} \nabla_{\sigma} R_{\rho\lambda} u^{\tau} \nabla_{\tau} R^{\rho}_{\mu} \quad (24)$$

is also a future tensor, non-vanishing unless  $\nabla_{[\mu} R_{\nu]\rho} = 0$ . But this means<sup>||</sup>, (corollaries 2.1 or 2.7 in [6]) that for arbitrary timelike  $u^{\rho}$  the tensor

$$\overset{u}{T}_{\lambda\mu} \equiv u^{\sigma} \nabla_{\sigma} R_{\lambda\mu} \quad (25)$$

cannot be causal, neither future nor past, unless it takes the form  $k_{\mu} k_{\nu}$  for some null  $k^{\mu}$  possibly depending on  $\vec{u}$ . There arise two possibilities.

(i) If  $\overset{u}{T}_{\lambda\mu} = k_{\lambda} k_{\mu}$  for some  $\vec{u}$ , then (24) implies that  $T_{\alpha\beta\lambda\mu} u^{\alpha} u^{\beta} = 0$ , and this leads (corollary 2.3 in [6]) to  $T_{\alpha\beta\lambda\mu} = 0$  which is equivalent (property 3.4 in [73]) to  $\nabla_{[\nu} R_{\lambda]\mu} = 0$ . Thus, the tensor  $\nabla_{\nu} R_{\lambda\mu}$  is completely symmetric. Contracting (23) with  $u^{\alpha} v^{\beta}$ , where  $v^{\beta}$  is any other timelike vector, we derive

$$k_{\rho} \overset{v}{T}^{\rho}_{(\lambda} k_{\mu)} = 0 \implies k_{\rho} \overset{v}{T}^{\rho}_{\lambda} = 0$$

for any such  $\vec{v}$ . However, from (20) one deduces that, for arbitrary timelike  $\vec{w}$  and  $\vec{v}$

$$\overset{w}{T}_{\lambda\mu} \overset{v}{T}^{\lambda\mu} = w^{\sigma} \nabla_{\sigma} R_{\lambda\mu} v^{\tau} \nabla_{\tau} R^{\lambda\mu} = 0 \quad (26)$$

which together with the previous implies that, in fact, there is a fixed null 1-form  $k_{\mu}$  independent of  $u^{\rho}$  such that, using the complete symmetry of the tensor

$$\nabla_{\nu} R_{\lambda\mu} = f k_{\nu} k_{\lambda} k_{\mu}. \quad (27)$$

This reasoning has been performed at a fixed point. One cannot therefore extract conclusions by taking derivatives of the previous expression until the second possibility has been analyzed.

(ii) Suppose then that, at the given point, all the tensors (25) are non-causal, for arbitrary timelike  $\vec{u}$ . Furthermore, from

$$T_{\alpha\beta\lambda\mu} u^{\alpha} u^{\beta} v^{\lambda} v^{\mu} \geq 0$$

it follows that  $\overset{u}{T}_{\rho\lambda} v^{\lambda}$  is spacelike for arbitrary timelike future directed  $\vec{u}, \vec{v}$ . This immediately implies that  $\overset{u}{T}_{\rho\lambda}$  cannot be of algebraic type  $[1, 1 \dots 1]$  for any  $\vec{u}$  as this type has a timelike eigenvector. Similarly,  $\overset{u}{T}_{\rho\lambda}$  cannot be of type  $[z\bar{z}1 \dots 1]$  for any  $\vec{u}$ , because if it were one could easily check that the only way  $\overset{u}{T}_{\rho\lambda} v^{\lambda}$  is spacelike for arbitrary timelike future directed  $\vec{v}$  is that

$$\overset{u}{T}_{\mu\nu} = \nu (k_{\mu} k_{\nu} - \ell_{\mu} \ell_{\nu}) + \overset{u}{P}_{\mu\nu}, \quad \nu(u) \neq 0$$

<sup>||</sup> Observe the change of signature with respect to Ref.[6], which may be a little confusing. One should not use, for instance, proposition 2.1 in [6], but rather the corollary 2.1 in that paper.

for null  $k_\mu$  and  $\ell_\nu$ —possibly depending on  $\vec{u}$ —such that  $k^\mu \ell_\mu = -1$  and with  $\overset{u}{P}_{\mu\nu} = \overset{u}{P}_{(\mu\nu)}$ ,  $\overset{u}{P}_{\mu\nu} \ell^\mu = 0$ ,  $\overset{u}{P}_{\mu\nu} k^\mu = 0$ . But then  $\overset{u}{T}_{\mu\nu} \ell^\mu$  and  $\overset{u}{T}_{\mu\nu} k^\mu$  would be both null and

$$T_{\alpha\beta\lambda\mu} u^\alpha u^\beta \ell^\lambda \ell^\mu = 0, \quad T_{\alpha\beta\lambda\mu} u^\alpha u^\beta k^\lambda k^\mu = 0$$

which would imply (property 2.3 in [6]) that in fact

$$T_{\alpha\beta\lambda\mu} \ell^\lambda \ell^\mu = 0, \quad T_{\alpha\beta\lambda\mu} k^\lambda k^\mu = 0.$$

From here one would easily deduce that, for all timelike  $\vec{u}$ ,  $\overset{u}{T}_{\mu\nu} \ell^\mu$  would be null and proportional to  $k_\nu$  and  $\overset{u}{T}_{\mu\nu} k^\mu$  would be null and proportional to  $\ell_\nu$ , with both  $k_\nu$  and  $\ell_\nu$  independent of  $u^\mu$ . This would lead to

$$\nabla_\rho R_{\mu\nu} = w_\rho (k_\mu k_\nu - \ell_\mu \ell_\nu) + P_{\rho\mu\nu}$$

for fixed  $w_\rho$  and null  $k_\mu$  and  $\ell_\mu$  with  $P_{\rho\mu\nu} = P_{\rho(\mu\nu)}$  and  $P_{\rho\mu\nu} \ell^\mu = 0$ ,  $P_{\rho\mu\nu} k^\mu = 0$ . Using here that  $\nabla_\rho R^\rho{}_\nu = 0$  and (20) this would inevitably lead to  $\nabla_\nu R_{\lambda\mu} = 0$ .

The only remaining possibility is that all tensors (25) have algebraic type  $[21 \dots 1]$  or  $[31 \dots 1]$ , in both cases having a null eigenvector  $k_\mu$ —possibly depending on  $\vec{u}$ . The former type  $[21 \dots 1]$  is easily ruled out by using again the property (20) which leads to  $\overset{u}{T}_{\lambda\mu} = k_\lambda k_\mu$  for null  $k_\mu$ , that is, to the already considered possibility (i). Let us finally consider the remaining type  $[31 \dots 1]$ . With the help once again of (20) it follows that  $\overset{u}{T}_{\mu\nu} = k_{(\mu} \overset{u}{q}_{\nu)}$  with  $\overset{u}{q}_\mu$  spacelike and orthogonal to  $k_\mu$ . But then,  $T_{\alpha\beta\lambda\mu} u^\alpha u^\beta k^\lambda = 0$  so that  $T_{\alpha\beta\lambda\mu} k^\lambda = 0$  and using the same type of reasonings as before  $\overset{u}{T}_{\mu\nu} = k_{(\mu} \overset{u}{q}_{\nu)}$  with a fixed null  $k_\mu$  which does not depend on  $u^\rho$ . This implies that

$$\nabla_\nu R_{\lambda\mu} = P_{\nu(\lambda} k_{\mu)}$$

for some  $P_{\nu\lambda}$  with  $k^\lambda P_{\nu\lambda} = 0$ . Using now once more the several formulas in (20) one derives the result that, in fact,

$$\nabla_\nu R_{\lambda\mu} = k_\nu q_{(\lambda} k_{\mu)}$$

with  $q^\mu$  spacelike and orthogonal to  $k_\mu$ .

Summarizing the two possibilities (i) and (ii), at every point of the domain  $D$  the covariant derivative of the Riemann tensor must take the form

$$\nabla_\nu R_{\lambda\mu} = k_\nu p_{(\lambda} k_{\mu)}$$

where  $k_\mu$  is a differentiable null vector field and  $p^\mu$  is a differentiable vector field orthogonal to  $k_\mu$ —it may be proportional to  $k_\mu$  in some places, or even zero—. But then

$$T_{\alpha\beta\lambda\mu} = (p^\rho p_\rho) k_\alpha k_\beta k_\lambda k_\mu$$

so that, as  $T_{\alpha\beta\lambda\mu}$  is a parallel tensor from its definition, it follows that there is a vector proportional to  $k^\mu$  which is parallel, unless  $p_\mu$  is null or zero. But in these cases  $T_{\alpha\beta\lambda\mu} = 0$  implying  $\nabla_{[\nu} R_{\lambda]\mu} = 0$  and therefore (27) holds. But then again, as  $\nabla_\alpha \nabla_\nu R_{\lambda\mu} = 0$ , it easily follows that, if  $f$  is non-zero, then a null vector field proportional to  $k_\mu$  is covariantly constant.

To finalize, as the existence of a null covariantly constant vector field is against the hypothesis of the theorem, it necessarily follows

$$\nabla_\nu R_{\lambda\mu} = 0.$$

Therefore, the Ricci tensor is a parallel symmetric tensor field, so that from Lemma 3.1 the manifold is either non-degenerately decomposable or the Ricci tensor is proportional to the metric. By decomposing

into irreducible parts, all of them 2-symmetric as has been already explained several times, the only relevant cases are that of Einstein spaces (for the reduced spaces, if necessary)

$$R_{\lambda\mu} = \frac{R}{m} g_{\lambda\mu}.$$

(Here  $m \leq n$  is the dimension of the given irreducible submanifold if this is necessary, and then  $g_{\lambda\mu}$ ,  $R_{\lambda\mu}$  and  $R$  stand for its corresponding reduced metric tensor, Ricci tensor and scalar curvature, respectively.) However, from this relation and the contracted Bianchi identity —applied to the reduced parts— one deduces

$$\nabla_\rho R^\rho{}_{\beta\gamma\delta} = 0,$$

hence, contracting  $\nu$  with  $\lambda$  in the equation (9) applied to the corresponding Riemann tensor, it follows that

$$R \nabla_\mu R_{\alpha\beta\gamma\delta} = 0$$

so that finally each of the irreducible components of the spacetime is either locally symmetric or its Ricci tensor vanishes, which finishes the proof.  $\blacksquare$

**Remark.** It must be stressed that this proof is only valid for Lorentzian manifolds, as the definition of future tensors requires this signature. I would like to stress, in passing, that this proof shows the mathematical potentialities of the superenergy construction and of the theory of causal tensors.

Now, one can at last prove that the narrow space left between locally symmetric and 2-symmetric Lorentzian manifolds can only be filled by spaces with a covariantly constant null vector field. In order to alleviate the path to the final result, we first present the algebraic part of the reasoning as separate lemmas, interesting on their own.

**Lemma 4.1** *At any point of a Ricci-flat Lorentzian manifold  $(\mathcal{V}, g)$  where the Riemann (or equivalently the Weyl) tensor satisfies the first in (16)*

$$\nabla_{(\tau} C^\rho{}_{\nu)\lambda\mu} \nabla_\rho C_{\alpha\beta\gamma\delta} = 0 \tag{28}$$

*there exists a null vector  $k^\mu$  such that*

$$k^\rho \nabla_\rho C_{\alpha\beta\gamma\delta} = 0, \quad k^\alpha \nabla_\alpha C_{\alpha\beta\gamma\delta} = 0.$$

*Proof.* First of all, there must be at least a tangent vector  $p^\mu$  such that

$$p^\rho \nabla_\rho C_{\alpha\beta\gamma\delta} = 0$$

as otherwise it would follow from (28) that  $\nabla_{(\tau} C^\rho{}_{\nu)\lambda\mu} = 0$  which together with the identity  $\nabla_{[\tau} C_{\rho\nu]\lambda\mu} = 0$  would lead to  $\nabla_\tau C_{\rho\nu\lambda\mu} = 0$  anyway. Let us define, for any arbitrary 2-form  $F_{\mu\nu}$ , the following tensor

$${}^F \Omega_{\rho\alpha\beta} \equiv \nabla_\rho C_{\alpha\beta\gamma\delta} F^{\gamma\delta}$$

which has the immediate properties

$${}^F \Omega_{\rho\alpha\beta} = {}^F \Omega_{\rho[\alpha\beta]}, \quad {}^F \Omega_{[\rho\alpha\beta]} = 0 \quad {}^F \Omega^\rho{}_{\rho\beta} = 0.$$

The main condition (28) can then be reexpressed as, for *any* 2-forms  $F^{\mu\nu}$  and  $G^{\mu\nu}$ ,

$${}^F \Omega_{(\tau}{}^\rho{}_{\nu)} {}^G \Omega_{\rho\alpha\beta} = 0. \tag{29}$$

If the vector  $p^\mu$  is non-null, it follows directly by orthogonal decomposition

$${}^F \Omega_{\rho\alpha\beta} = {}^F T_{\rho\alpha\beta} + 2 {}^F C_{\rho[\beta} p_{\alpha]} \tag{30}$$



where

$$\begin{aligned} \overset{F}{T}_{\rho\alpha\beta} &= \overset{F}{T}_{\rho[\alpha\beta]}, \quad \overset{F}{T}_{[\rho\alpha\beta]} = 0, \quad \overset{F}{T}{}^\rho{}_{\rho\beta} = 0, \quad \overset{F}{C}_{\rho\beta} = \overset{F}{C}_{(\rho\beta)}, \quad \overset{F}{C}{}^\rho{}_\rho = 0, \\ \overset{F}{T}_{\rho\alpha\beta} p^\rho &= 0, \quad \overset{F}{T}_{\rho\alpha\beta} p^\alpha = 0, \quad \overset{F}{C}_{\rho\beta} p^\rho = 0. \end{aligned}$$

But then, it is very easy to see that (29) implies the following conditions for arbitrary  $F^{\mu\nu}$  and  $G^{\mu\nu}$

$$\begin{aligned} \overset{F}{C}{}^\tau{}_\rho \overset{G}{C}{}_{\rho\alpha} &= 0, \\ \overset{F}{T}{}_{(\tau}{}^\rho{}_{\nu)} \overset{G}{C}{}_{\rho\alpha} &= 0, \quad \overset{F}{C}{}^\tau{}_\rho \overset{G}{T}{}_{\rho\alpha\beta} = 0, \quad \implies \quad \overset{F}{C}{}^\tau{}_\rho \overset{G}{T}{}_{\alpha\rho\beta} = 0 \\ \overset{F}{T}{}_{(\tau}{}^\rho{}_{\nu)} \overset{G}{T}{}_{\rho\alpha\beta} &= 0 \end{aligned}$$

where the implication in the middle line follows on using the property  $\overset{F}{T}_{[\rho\alpha\beta]} = 0$ . The first of these conditions implies necessarily (by repeated application of corollary 2.7 in [6]) that

$$\overset{F}{C}{}_{\rho\alpha} = f(F) k_\rho k_\alpha$$

for a *fixed* null vector  $k_\mu$  independent of  $F \in \Lambda_2$  and a function—which may have zeros—depending on  $F \in \Lambda_2$ . But then the second condition implies that, for any  $F, G \in \Lambda_2$ ,

$$f(F) \overset{G}{T}{}_{\rho\alpha\beta} k^\rho = 0, \quad f(F) \overset{G}{T}{}_{\rho\alpha\beta} k^\alpha = 0$$

so that either  $f(F) = 0$  or all the tensors  $\overset{G}{T}{}_{\rho\alpha\beta}$  are completely orthogonal to  $k^\rho$  for arbitrary  $G \in \Lambda_2$ . In the latter case, it follows that  $k^\rho \nabla_\rho C_{\alpha\beta\gamma\delta} = 0$ . In the former case, all tensors  $\overset{F}{C}{}_{\rho\alpha}$  vanish and therefore one can deduce, by using the symmetry and trace properties of  $\nabla_\rho C_{\alpha\beta\gamma\delta}$ , that this tensor is orthogonal in all its indices to  $p^\mu$ . But then one can start again the reasoning now restricted to the tangent subspace  $\{\vec{p}\}^\perp$  by choosing another  $\vec{p}' \in \{\vec{p}\}^\perp$  such that  $p'^\rho \nabla_\rho C_{\alpha\beta\gamma\delta} = 0$ , and so on until either there is a null  $k^\mu$  such that  $k^\rho \nabla_\rho C_{\alpha\beta\gamma\delta} = 0$  or the tensor  $\nabla_\rho C_{\alpha\beta\gamma\delta}$  vanishes.

All in all, the conclusion is that we could have assumed, from the beginning, that  $p^\rho = k^\rho$  is null. In this case, one can directly write, analogously to (30),

$$\overset{F}{\Omega}_{\rho\alpha\beta} = \overset{F}{S}_{\rho\alpha\beta} + 2 \overset{F}{A}_{\rho[\beta} \ell_{\alpha]}$$

where  $\ell_\mu$  is null such that  $k^\mu \ell_\mu = -1$  and

$$\begin{aligned} \overset{F}{S}_{\rho\alpha\beta} &= \overset{F}{S}_{\rho[\alpha\beta]}, \quad \overset{F}{S}_{[\rho\alpha\beta]} = 0, \quad \overset{F}{A}_{\rho\beta} = \overset{F}{A}_{(\rho\beta)}, \quad \overset{F}{A}{}^\rho{}_\rho = 0, \\ \overset{F}{S}{}^\rho{}_{\rho\beta} + \overset{F}{A}_{\rho\beta} \ell^\rho &= 0, \quad \overset{F}{S}_{\rho\alpha\beta} k^\rho = 0, \quad \overset{F}{S}_{\rho\alpha\beta} k^\alpha = 0, \quad \overset{F}{A}_{\rho\beta} k^\rho = 0. \end{aligned}$$

The main condition (29) splits then into

$$\begin{aligned} \overset{F}{A}{}^\tau{}_\rho \overset{G}{A}_{\rho\beta} &= 0, \\ \overset{F}{A}{}^\tau{}_\rho \overset{G}{S}_{\rho\alpha\beta} &= 0, \\ \overset{F}{S}{}_{(\tau}{}^\rho{}_{\nu)} \overset{G}{A}_{\rho\beta} + \overset{F}{A}_{\tau\nu} \overset{G}{A}_{\rho\beta} \ell^\rho &= 0, \\ \overset{F}{S}{}_{(\tau}{}^\rho{}_{\nu)} \overset{G}{S}_{\rho\alpha\beta} + \overset{F}{A}_{\tau\nu} \overset{G}{S}_{\rho\alpha\beta} \ell^\rho &= 0 \end{aligned}$$

for arbitrary 2-forms  $F^{\mu\nu}$  and  $G^{\mu\nu}$ . The first of these provides, as before,

$$\overset{F}{A}_{\rho\alpha} = f(F) k_\rho k_\alpha$$

but then the third leads to

$$f(G) \overset{F}{A}_{\tau\nu} = 0$$

which in any case implies

$$\overset{F}{A}_{\tau\nu} = 0, \quad \forall F \in \Lambda_2.$$

But then one deduces that for arbitrary  $F \in \Lambda_2$ ,  $\overset{F}{\Omega}_{\rho\alpha\beta} = \overset{F}{S}_{\rho\alpha\beta}$  so that in fact not only  $k^\rho \nabla_\rho C_{\alpha\beta\gamma\delta} = 0$  but also

$$k^\alpha \nabla_\rho C_{\alpha\beta\gamma\delta} = 0$$

as required. ■

**Lemma 4.2** *At any point of a Ricci-flat 2-symmetric Lorentzian manifold  $(\mathcal{V}, g)$  where the Riemann (or equivalently the Weyl) tensor satisfies (28) there exists a null vector  $k^\mu$  such that*

$$\nabla_\rho C_{\alpha\beta\gamma\delta} = 4k_\rho k_{[\alpha} B_{\beta][\delta} k_{\gamma]} \quad (31)$$

where  $B_{\beta\mu}$  is a symmetric tensor with the following properties

$$k^\mu B_{\beta\mu} = 0, \quad B^\mu{}_\mu = 0. \quad (32)$$

*Proof.* From the previous lemma, we know that, for arbitrary tangent vector  $\vec{v}$ , the tensor  $\nabla_{\vec{v}} C_{\alpha\beta\gamma\delta}$  satisfies

$$\nabla_{\vec{v}} C_{\alpha\beta\gamma\delta} k^\alpha = 0$$

which, with the essential help of (21) (in particular using  $\nabla_{\vec{v}} C_{\alpha\beta\gamma\delta} \nabla_{\vec{v}} C^{\alpha\beta\gamma\delta} = 0$ ) can be shown to imply in our case¶ [64]

$$\nabla_{\vec{v}} C_{\alpha\beta\gamma\delta} = 4k_{[\alpha} \overset{v}{B}_{\beta][\delta} k_{\gamma]} + 2k_{[\alpha} \overset{v}{D}_{\beta]\gamma\delta} + 2k_{[\gamma} \overset{v}{D}_{\delta]\alpha\beta}$$

where

$$\begin{aligned} \overset{v}{B}_{\beta\mu} &= \overset{v}{B}_{(\beta\mu)}, \quad \overset{v}{B}_{\beta\mu} k^\mu = 0, \quad \overset{v}{B}^\mu{}_\mu = 0, \\ \overset{v}{D}_{\beta\lambda\mu} &= \overset{v}{D}_{\beta[\lambda\mu]}, \quad \overset{v}{D}_{[\beta\lambda\mu]} = 0, \quad k^\beta \overset{v}{D}_{\beta\lambda\mu} = 0, \quad k^\mu \overset{v}{D}_{\beta\lambda\mu} = 0, \quad \overset{v}{D}^\rho{}_{\rho\mu} = 0 \end{aligned}$$

and, without loss of generality, one can choose another null vector  $\ell_\mu$  such that  $\overset{v}{B}_{\beta\mu} \ell^\mu = 0$ ,  $\ell^\beta \overset{v}{D}_{\beta\lambda\mu} = 0$  and  $\ell^\mu \overset{v}{D}_{\beta\lambda\mu} = 0$  too. This leads to

$$\nabla^\rho C_{\alpha\beta\gamma\delta} = 4k_{[\alpha} B^\rho{}_{\beta][\delta} k_{\gamma]} + 2k_{[\alpha} D^\rho{}_{\beta]\gamma\delta} + 2k_{[\gamma} D^\rho{}_{\delta]\alpha\beta}$$

for tensors  $B_{\rho\beta\mu}$  and  $D_{\rho\beta\lambda\mu}$  with the properties

$$\begin{aligned} B_{\rho\beta\mu} &= B_{\rho(\beta\mu)}, \quad k^\rho B_{\rho\beta\mu} = 0, \quad k^\beta B_{\rho\beta\mu} = 0, \quad B^\rho{}_{\rho\mu} = 0, \quad B_{\beta}{}^\rho{}_\rho = 0, \quad D_{\rho\beta\lambda\mu} = D_{\rho\beta[\lambda\mu]}, \\ D_{\rho[\beta\lambda\mu]} &= 0, \quad k^\rho D_{\rho\beta\lambda\mu} = 0, \quad k^\beta D_{\rho\beta\lambda\mu} = 0, \quad k^\mu D_{\rho\beta\lambda\mu} = 0, \quad D_{\rho}{}^\sigma{}_\sigma\mu = 0, \quad D^\rho{}_{\rho\lambda\mu} = 0, \quad D^\rho{}_{\beta\rho\mu} = 0 \end{aligned}$$

and, without loss of generality  $B_{\rho\beta\mu} \ell^\mu = 0$ ,  $D_{\rho\beta\lambda\mu} \ell^\beta = 0$  and  $D_{\rho\beta\lambda\mu} \ell^\mu = 0$  too. But then, using the last in (21) one easily obtains

$$D^{\rho\beta\lambda\mu} D_{\rho\beta\lambda\mu} = 0$$

which implies necessarily

$$D_{\rho\beta\lambda\mu} = k_\rho A_{\beta\lambda\mu}$$

for some  $A_{\beta\lambda\mu}$  with the necessary properties. Splitting the other tensor as follows

$$B_{\rho\beta\mu} = k_\rho B_{\beta\mu} + \tilde{B}_{\rho\beta\mu}, \quad \ell^\rho \tilde{B}_{\rho\beta\mu} = 0$$

¶ Following the standard nomenclature in General Relativity [75], a Weyl-like tensor with this form is said to be of ‘type III or N’, see e.g. [23, 24, 64] and references therein.

one can prove, by using  $\nabla_{[\rho} C_{\alpha\beta]\gamma\delta} = 0$ , that in fact

$$A_{\beta\lambda\mu} = 2\tilde{B}_{[\lambda\mu]\beta}.$$

The basic equation (28) can now be recalculated leading to

$$\tilde{B}^\rho{}_{\beta\delta} B_{\rho\mu} = 0, \quad \tilde{B}^\rho{}_{\beta\delta} (\tilde{B}_{\rho\mu\nu} - 2\tilde{B}_{\mu\nu\rho}) = 0.$$

It is an exercise to check that, for a spatial tensor such as  $\tilde{B}_{\rho\mu\nu}$ —which belongs to the tensor algebra constructed over the vector space  $\langle \vec{k}, \vec{\ell} \rangle^\perp$ —the last condition implies

$$\tilde{B}_{\rho\mu\nu} = 0$$

which leads finally to the result (31) with (32). ■

**Theorem 4.2** *Let  $D \subset \mathcal{V}$  be a simply connected domain of an  $n$ -dimensional 2-symmetric Lorentzian manifold  $(\mathcal{V}, g)$ . Then, if there is no null covariantly constant vector field on  $D$ ,  $(D, g)$  is in fact locally symmetric.*

*Proof.* From the theorem (4.1) one only has to consider the case of Ricci-flat manifolds:

$$R_{\mu\nu} = 0, \quad R_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} \tag{33}$$

so that one can indistinctly use the Weyl or the Riemann tensor, while the Ricci tensor has to be set equal to zero. From the previous lemma 4.2 it follows that it must exist a null vector field  $k^\mu$  and a symmetric tensor field  $B_{\beta\mu}$  on  $D$  such that (31) and (32) hold. It follows that the tensor

$$T_{\alpha\beta\lambda\mu\tau\nu} \equiv 4\nabla_\alpha C_{\lambda\rho\tau\sigma} \nabla_\beta C_\mu{}^\rho{}_\nu{}^\sigma = 4(B^{\rho\sigma} B_{\rho\sigma}) k_\alpha k_\beta k_\lambda k_\mu k_\tau k_\nu \tag{34}$$

(this is in fact, in this case, the (super)<sup>2</sup>-energy tensor, formula (44) in [73]). As  $T_{\alpha\beta\lambda\mu\tau\nu}$  is parallel by assumption, it follows that in any region where  $B_{\mu\nu}$  is not zero there exists a parallel null vector field proportional to  $k_\mu$ . ■

The main results can be summarized in the following central theorem.

**Theorem 4.3** *Let  $D \subset \mathcal{V}$  be a simply connected domain of an  $n$ -dimensional 2-symmetric Lorentzian manifold  $(\mathcal{V}, g)$ . Then, the line element on  $D$  is (possibly a flat extension of) the direct product of a certain number of locally symmetric proper Riemannian manifolds times either*

- (i) *a Lorentzian locally symmetric spacetime (in which case the whole  $(D, g)$  is locally symmetric), or*
- (ii) *a Lorentzian manifold with a covariantly constant null vector field —hence belonging to the general Brinkmann's class presented below in (35).*

*Proof.* It follows from Lemma 3.1 and the previous theorems 4.1 and 4.2. ■

**Remarks:**

- (i) Of course, the number of proper Riemannian symmetric manifolds can be zero, so that the whole 2-symmetric spacetime, if not locally symmetric, is given just by a line-element of the form (35).
- (ii) Although mentioned explicitly for the sake of clarity, it is obvious that the block added in any flat extension can also be considered as a particular case of a locally symmetric part building up the whole space.
- (iii) This theorem provides a full characterization of the 2-symmetric spaces using the classical results on the symmetric ones: their original classification (for the semisimple case) was given in [18], see also [5, 47], and the general problem was solved for Lorentzian signature in [16]. Combining these results with those for proper Riemannian metrics [17, 18, 47], a complete classification is achieved.

The most general local line-element for a Lorentzian manifold with covariantly constant null vector field was discovered by Brinkmann [10] by studying the Einstein spaces which can be mapped conformally to each other. In appropriate local coordinates

$$\{x^0, x^1, x^i\} = \{u, v, x^i\}, \quad (i, j, k, \dots = 2, \dots, n-1)$$

the line-element reads (see also [89])

$$ds^2 = -2du(dv + Hdu + W_i dx^i) + g_{ij} dx^i dx^j \quad (35)$$

where the functions  $H$ ,  $W_i$  and  $g_{ij} = g_{ji}$  are independent of  $v$ , otherwise arbitrary, and the parallel null vector field is given by

$$k_\mu dx^\mu = -du, \quad k^\mu \partial_\mu = \partial_v. \quad (36)$$

Given that all 2-symmetric non-symmetric Lorentzian manifolds contain a covariantly constant null vector field according to theorems 4.2 and 4.3, and that the explicit local form (35) is known, it is a simple matter of calculation to identify which manifolds among (35) are actually 2-symmetric. Using theorem 4.3 and its remarks, this provides —by direct product with proper Riemannian symmetric manifolds if adequate— all possible *non-symmetric* 2-symmetric spacetimes. This is the subject of the second paper [74].

A particularly simple example of these 2-symmetric spacetimes is provided by the “plane waves” (see e.g. [35, 75]) for which<sup>+</sup>  $g_{ij} = \delta_{ij}$ ,  $W_i = 0$  and the remaining function is a quadratic expression on the coordinates  $x^i$

$$H = a_{ij}(u)x^i x^j. \quad (37)$$

The functions  $a_{ij}(u)$  are arbitrary for general plane waves, but in our case the 2-symmetry is easily seen to imply then that

$$\frac{d^2 a_{ij}}{du^2} = 0, \implies a_{ij}(u) = c_{ij}u + b_{ij}$$

for some constants  $c_{ij}, b_{ij}$ . These spaces have been actually identified recently in [60]. Note that one can add any number of proper Riemannian locally symmetric spaces to this one keeping the whole 2-symmetry.

## 5. Future work and open problems

Arguably, the decisive step to characterize all  $k$ -symmetric spacetimes has been achieved with theorems 4.1, 4.2 and 4.3. The same or even more modest techniques —for instance, using repeatedly the Ricci identity— will allow for a complete solution to the problem with arbitrary  $k$  now that the 2nd order stage has been solved. As a matter of fact, it seems plausible that the following conjecture will hold

**Conjecture 1** *Let  $D \subset \mathcal{V}$  be a simply connected domain of an  $n$ -dimensional  $k$ -symmetric Lorentzian manifold  $(\mathcal{V}, g)$ . Then, if there is no null covariantly constant vector field on  $D$ ,  $(D, g)$  is in fact locally symmetric.*

If this is true, then a result similar to theorem 4.3 will be valid and the  $k$ -symmetric spacetimes will be just the direct sum of lower-order symmetric ones plus a particular  $k$ -symmetric one of the Brinkmann class (35). Actually, there are obvious  $k$ -symmetric spaces within the plane waves: just use the same form (37) for the function  $H$  but let the functions  $a_{ij}(u)$  be polynomials of degree up to  $k-1$ , see also [60].

As a matter of fact, it might be the case that not only  $k$ -symmetric, but also all  $k$ -recurrent spacetimes, have the same type of hierarchy, see also [25]. I would like to put forward the second conjecture:

<sup>+</sup> Actually, in  $n = 4$ , all 2-recurrent spacetimes were given in [54], see also [82, 83, 84] where they were proved to be a subset of the Brinkmann class (35), and the conformally 2-recurrent spacetimes belonging to (35) and satisfying  $W_i = 0$ ,  $g_{ij} = \delta_{ij}$ , were all found in [25].

**Conjecture 2** *Let  $D \subset \mathcal{V}$  be a simply connected domain of an  $n$ -dimensional  $k$ -recurrent Lorentzian manifold  $(\mathcal{V}, g)$ . Then, if there is no null recurrent vector field on  $D$ ,  $(D, g)$  is in fact recurrent.*

In this sense, it should be interesting to know if there is any characterization of the spacetimes which belong to the Brinkmann class—that is, they have a parallel null vector field—or to the more general classes with a recurrent null vector field, see e.g. [52], but are neither recurrent nor symmetric of any order, if they exist.

Of course, all the above can be generalized, if desired, to conformally  $k$ -symmetric, or  $k$ -recurrent, and Ricci  $k$ -symmetric and  $k$ -recurrent.

There are several other routes open and worth exploring. To start with, this paper has concentrated on Lorentzian signature, but the result does not apply, in principle, to other signatures. Mathematically, the resolution of the second-order symmetric spaces will not be complete until these other cases are fully solved.

Another interesting problem is that of semi-symmetric spacetimes (and with other signatures). This problem, as well as the conformally semi-symmetric one, can be solved in 4-dimensional Lorentzian manifolds by using spinors [62], see the classification in [44]. However, as happened with the case treated herein, the higher dimensional cases need further exploration.

Along the paper some relationships with holonomy, parallel tensors and vectors, or scalar invariants have appeared. In this sense, it would be interesting to know the list of all spacetimes with *constant* curvature scalar invariants [22]. Similarly, the implications of the existence of parallel tensors with rank higher than 2 would be of enormous help in this and related studies—apart from interesting on its own right—.

Finally, the algebraic solution of equations such as (13), or more interestingly, such as (8), (14) and (15) in general dimension  $n$  would be of great help.

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